Reliable Computing I

Lecture 6: Information Redundancy

Instructor: Mehdi Tahoori

INSTITUTE OF COMPUTER ENGINEERING (ITEC) – CHAIR FOR DEPENDABLE NANO COMPUTING (CDNC)

KIT – University of the State of Baden-Wuerttemberg and
National Research Center of the Helmholtz Association

www.kit.edu

Today's Lecture

- Code, codeword, binary code
- Error detecting and correcting codes
- Hamming distance and codes
- Parity prediction
  - Odd/even parity
  - Basic parity approaches

KIT – University of the State of Baden-Wuerttemberg and
National Laboratory of the Helmholtz Association

Error Detection through Encoding

- At logic level, codes provide means of masking or detecting errors
- Formally, code is a subset S of universe U of possible vectors
- A noncode word is a vector in set U-S

X1 is a codeword
<10010011>
due to multiple bit error, becomes
X3 = <10011100>
not detectable

X2 is a codeword, becomes X4 noncode
detectable

U
S = even parity
X1
X2
X3
X4
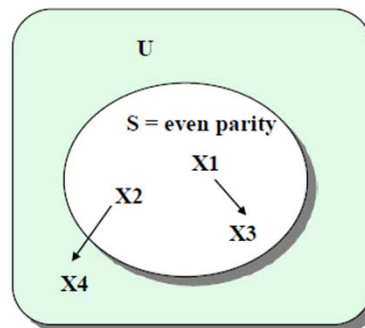
Basic Idea

- Start with k-bit data word
- Add r check bits
- Total = n-bit codeword (n=k+r)
- Map 2^k data words to 2^n sized codeword space
- Overhead = r/n (sometimes computed as r/k)
  - E.g., for (single-bit) parity, the overhead is 1/n

KIT – University of the State of Baden-Wuerttemberg and
National Laboratory of the Helmholtz Association

Basic Concepts
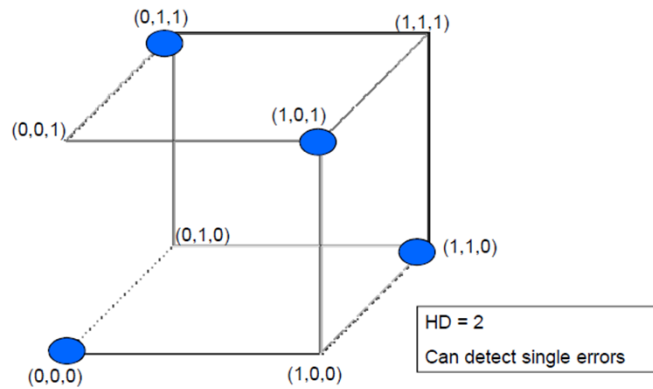
- Code, codeword, encoding, decoding error detection code, error correcting code
- Hamming distance properties:
  - The Hamming weight of a vector x (e.g., codeword), w(x), is number of nonzero elements of x.
  - Hamming distance between two vectors x and y, d(x,y) is number of bits in which they differ.
  - Distance of a code is a minimum of Hamming distances between all pairs of code words.
  - Example: x = (1011), y = (0110)

    w(x) = 3, w(y) = 2, d(x, y) = 3
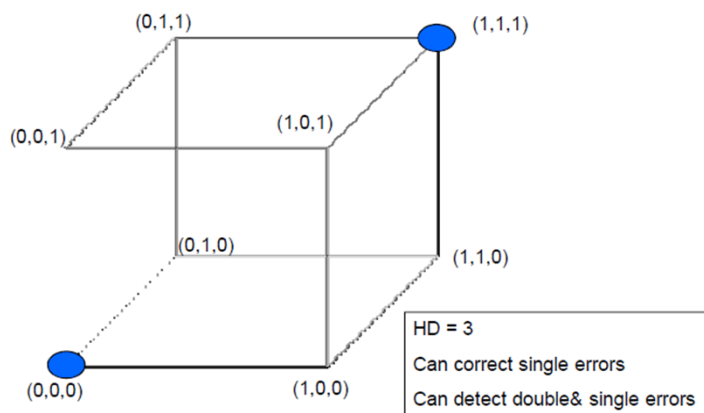
Hamming Distance

- Hamming distance (HD): number of bits in which two words differ from each other
  - E.g., 0010 and 1110 have a Hamming distance of ??
- For a group of codewords, the minimum HD between any two codewords determines the code's ability to detect and/or correct errors
  - This is a fundamental rule, not just some ad-hoc reasoning

Hamming Distance Visual: HD=2

(0,1,1) (1,1,1)

(0,0,1) (1,0,1)

(0,1,0) (1,1,0)

(0,0,0) (1,0,0)

HD = 2
Can detect single errors

Hamming Distance Visual: HD=3

(0,1,1) (1,1,1)

(0,0,1) (1,0,1)

(0,1,0) (1,1,0)

(0,0,0) (1,0,0)

HD = 3
Can correct single errors
Can detect double& single errors

KIT – University of the State of Baden-Wuerttemberg and
National Laboratory of the Helmholtz Association

Hamming Distance and Error Detection

■ Can detect up to t-bit errors if HD >= t + 1

(0,1,1)　　　　　(1,1,1)

(0,0,1)　　(1,0,1)

　　　　　(1,1,0)

(0,1,0)

(0,0,0)　　(1,0,0)
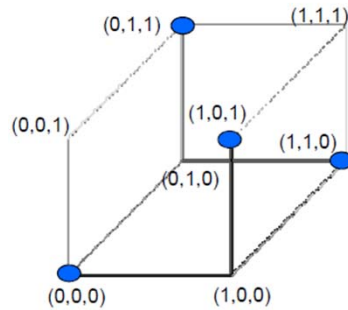
- **What if we receive 111?**
  - Could've been **011**
  - Could've been **101**
  - Could've been **110**
  - What about **001**? Or **000**??
- **What if we receive 011?**
  - Could it have been **001**???

HD=2, detects 1-bit errors

(c) 2013, Mehdi Tahoori　　　Reliable Computing I: Lecture 6　　　9

Hamming Distance and Error Detection

■ Can detect up to t-bit errors if HD >= t + 1

(0,1,1)　　　　　(1,1,1)

(0,0,1)　　(1,0,1)

　　　　　(1,1,0)

(0,1,0)

(0,0,0)　　(1,0,0)
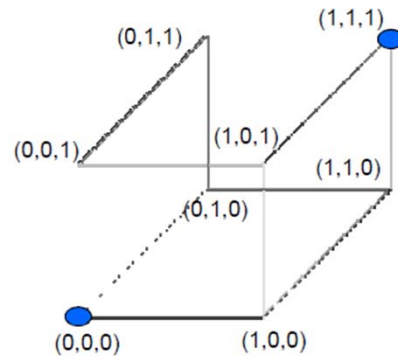
HD=3, detects 1,2-bit errors

(c) 2013, Mehdi Tahoori　　　Reliable Computing I: Lecture 6　　　10

KIT – University of the State of Baden-Wuerttemberg and
National Laboratory of the Helmholtz Association

Summary: Hamming Distance Properties

- To detect all error patterns of Hamming distance ≤ d,

  code distance must be ≥ d+1

  - e.g., code with distance 2 can detect patterns with distance 1 (i.e., single bit errors)

- To correct all error patterns of Hamming distance ≤ c,

  code distance must be ≥ 2c + 1

- To correct all patterns of Hamming distance c and detect up to d additional errors ,

  code distance must be ≥ 2c + d + 1

  - e.g., code with distance 3 can detect and correct all single-bit errors

Single-bit Parity

- Simplest error detection code
  - Adds one bit of redundancy to each data word
- Even (odd) parity: add bit such that total number of ones in codeword is even (odd)
  - E.g., 001010 gets a parity bit of 0 for even parity (1 for odd)
- Can detect all single-bit errors
  - Hamming distance ≥ 2
  - Could be greater than 2 if data words don't use all bit combinations
- Drawbacks:
  - Can't detect anything except single-bit errors

Parity Codes - Example

Odd and even parity codes for BCD data

Decimal Digit	BCD	BCD odd parity	BCD even parity
0	0000	00001	00000
1	0001	00010	00011
2	0010	00100	00101
3	0011	00111	00110
4	0100	01000	01001
5	0101	01011	01010
6	0110	01101	01100
7	0111	01110	01111
8	1000	10000	10001
9	1001	10011	10010
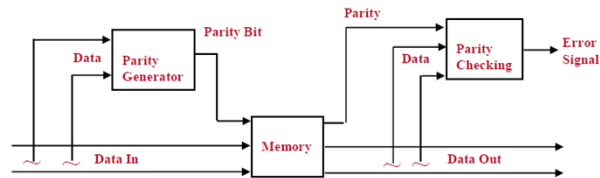
Organization of a memory that uses single-bit parity. The parity bit is generated when data is written to memory and checked when data is read.

XOR Tree for Parity Generation

Data Bits d_0, d_1, d_2, d_3, P — Generated Parity Bit, Error Signal
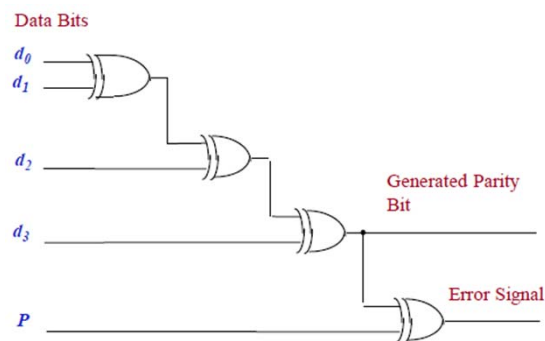
(c) 2013, Mehdi Tahoori

KIT – University of the State of Baden-Wuerttemberg and
National Laboratory of the Helmholtz Association

Codes for RAMs

Odd or Even

15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0   P   **Bit-per-word parity**

15 14 13 12 11 10 9 8 P_2 7 6 5 4 3 2 1 0 P_1 **Bit-per-byte parity**
Odd   Even

Chip 5 Chip 4 Chip 3 Chip 2 Chip 1
15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 P_4 P_3 P_2 P_1 **Bit-per-multiple-chips parity**

Chip 5 Chip 4 Chip 3 Chip 2 Chip 1
15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 P_4 P_3 P_2 P_1 **Bit-per-chip parity**

15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0 P_4 P_3 P_2 P_1 **Interlaced parity**
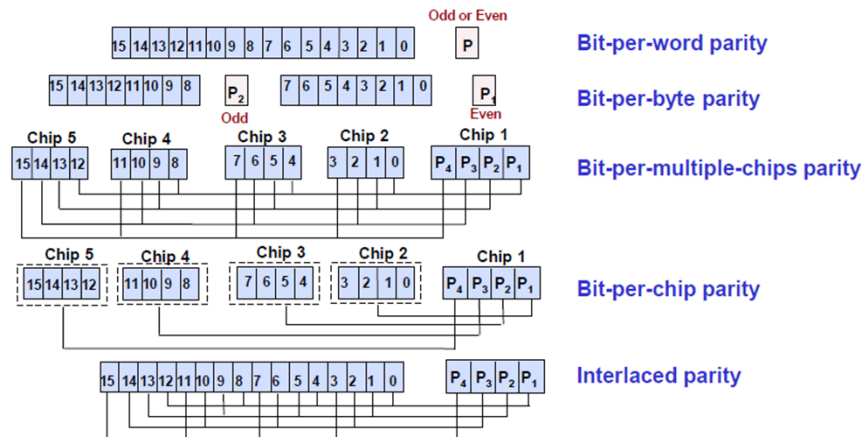
Parity Codes for Memory - Comparison

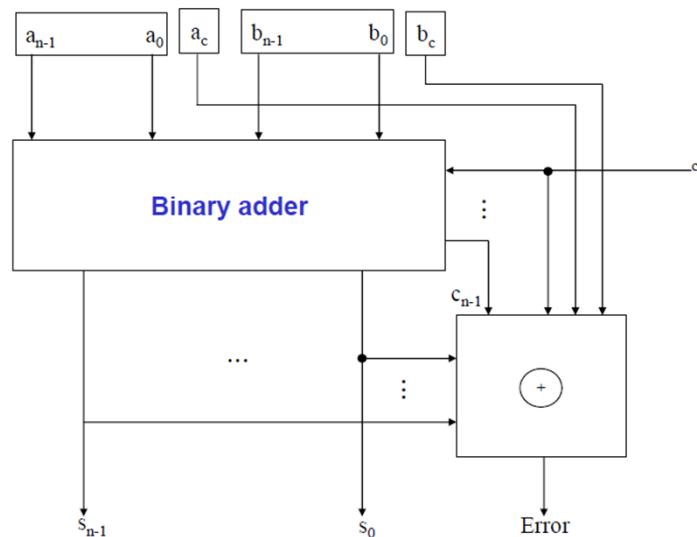| Parity Code	Advantages	Disadvantages
Bit-per-word: one parity bit per data word	Detects all single-bit errors	Certain errors undetected, e.g., a word, including parity bit becomes all 1s, due to a failure of a bus or a set of data buffers.
Bit-per-byte: each data portion (e.g., a byte) is protected by a separate parity bit; the parity of one group should be even and the parity of the other group should be odd	Detects all-1s and all-0s conditions	Ineffective for multiple errors, e.g., the whole-chip failure
Bit-per-multiple-chips: one bit from each chip is associated with a single parity bit	Detects failure of entire chip	Cannot locate failure of complete chip
Bit-per-chip: each parity bit is associated with one chip of the memory	Detects single-bit errors and identifies chip with erroneous bit	Susceptible to whole-chip failure, i.e., a single chip error can result in multiple bits to be corrupted and this may go undetected.
Interlaced: similar to the bit-per-multiple-chips; must ensure that no two adjacent bits are from the same parity group	Detects errors in adjacent bits	Parity groups are not based on physical organization of the memory

Parity Prediction in Arithmetic Circuits

- Binary Adder
  - Two inputs: $a = (a_{n-1} \ldots a_0 \, a_c)$ and $b = (b_{n-1} \ldots b_0 \, b_c)$
  - Two operands to be added: $(a_{n-1} \ldots a_0)$ and $(b_{n-1} \ldots b_0)$
  - a_c and b_c are check bits of a and b respectively
  - Encoded output will be $s = (s_{n-1} \ldots s_0 \, s_c)$ where
    $(s_{n-1} \ldots s_0)$ are determined by the ordinary binary
    addition of $(a_{n-1} \ldots a_0)$ to $(b_{n-1} \ldots b_0)$
    and s_c is the check bit for $(s_{n-1} \ldots s_0)$
  - Then $\quad s_c = \sum_{i=0}^{n-1} s_i \quad = \sum_{i=0}^{n-1} a_i \oplus \sum_{i=0}^{n-1} b_i \oplus \sum_{i=0}^{n-1} c_i$
  - Reduces to $\quad s_c = a_c \oplus b_c \oplus \sum_{i=0}^{n-1} c_i$

(c) 2013, Mehdi Tahoori   Reliable Computing I: Lecture 6   **19**

Parity Prediction in Binary Adder

(c) 2013, Mehdi Tahoori   Reliable Computing I: Lecture 6   **20**

KIT – University of the State of Baden-Wuerttemberg and
National Laboratory of the Helmholtz Association

Parity-Checked Binary Adder
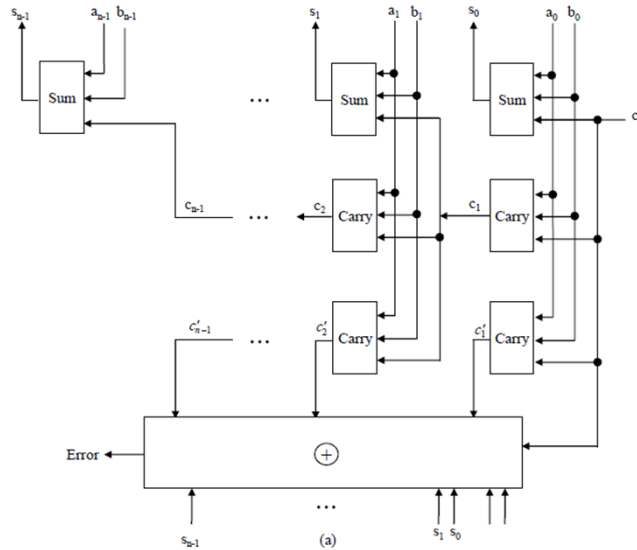
Binary Multiplier

$$p_0 = a_0 b_0$$
$$p_1 = a_0 b_1 \oplus a_1 b_0$$
$$p_2 = a_0 b_2 \oplus a_1 b_1 \oplus a_2 b_0 \oplus c_{1,1}$$
$$p_3 = a_0 b_3 \oplus a_1 b_2 \oplus a_2 b_1 \oplus a_3 b_0 \oplus c_{2,1} \oplus c_{1,2}$$
$$p_4 = a_1 b_3 \oplus a_2 b_2 \oplus a_3 b_1 \oplus c_{3,1} \oplus c_{2,2} \oplus c_{1,3}$$
$$p_5 = a_2 b_3 \oplus a_3 b_2 \oplus c_{3,2} \oplus c_{2,3} \oplus c_{1,4}$$
$$p_6 = a_3 b_3 \oplus c_{3,3} \oplus c_{2,4}$$
$$p_7 = c_{3,4}$$

■ Therefore, denoting the check bit for ($p_7 \ldots p_0$) by p_c

$$p_c = \sum_{i=0}^{7} p_i$$

$$= (\sum_{i=0}^{3} a_i)(\sum_{i=0}^{3} b_i) \oplus \sum_{i=1}^{3} \sum_{j=1}^{4} c_{i,j}$$

$$= a_c b_c \oplus \sum_{i=1}^{3} \sum_{j=1}^{4} c_{i,j}$$

KIT – University of the State of Baden-Wuerttemberg and
National Laboratory of the Helmholtz Association

Multiplier Using Array of Full-Half Adders
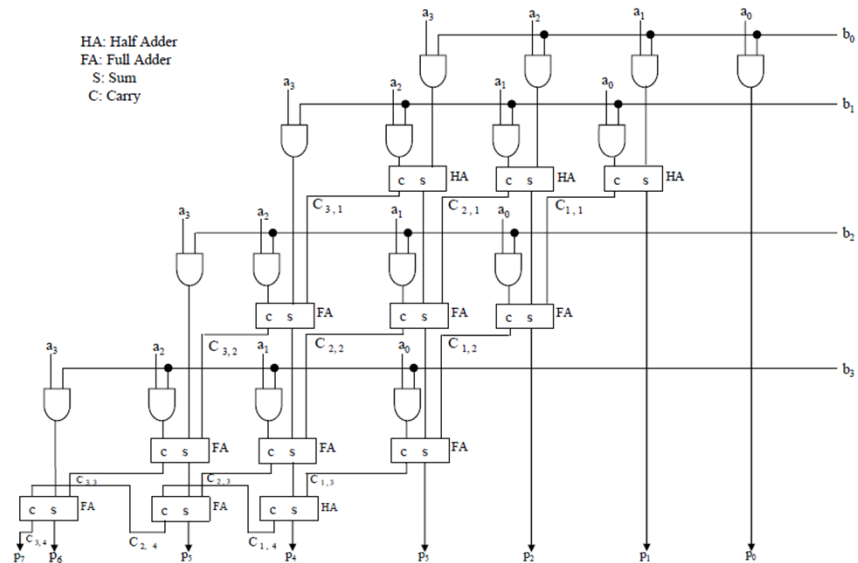
Parity Checked Multiplier
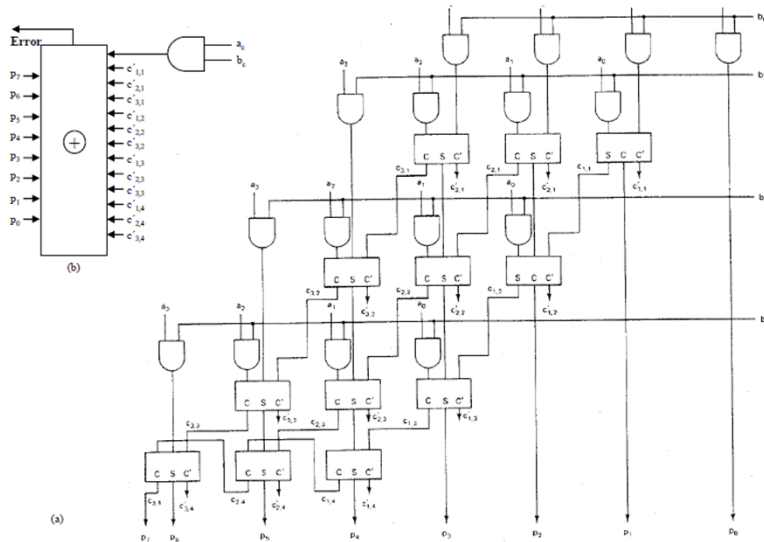
Overlapping Parity (for single-bit errors)

- Parity groups are formed with each bit appearing in more than one parity group
- Errors can be detected and located
- Erroneous bit can be corrected by a simple complementation

k=4 information bits — i3, i2, i1, i0

r=3 parity bits — p2, p1, p0

Which bit has error?	Parity bits affected
i3	p2, p1, p0
i2	p2, p1
i1	p2, p0
i0	p1, p0
p2	p2
p1	p1
p0	p0

When receiving codeword, re-compute 3 parity bits and compare to those that were sent. If different, can diagnose error (and correct it)!

Error Correction with Overlapped Parity

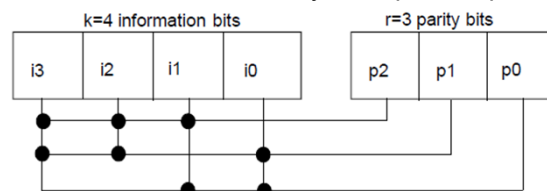
3-8 Decode

C 0 — Correct Bit 0
C 1 — Correct Bit 1
C 2 — Correct Bit 2
C 3 — Correct Bit 3
CP 0 — Correct Bit P_0
CP 1 — Correct Bit P_1
CP 2 — Correct Bit P_2
E — No Error

Corrected Bits

KIT – University of the State of Baden-Wuerttemberg and
National Laboratory of the Helmholtz Association
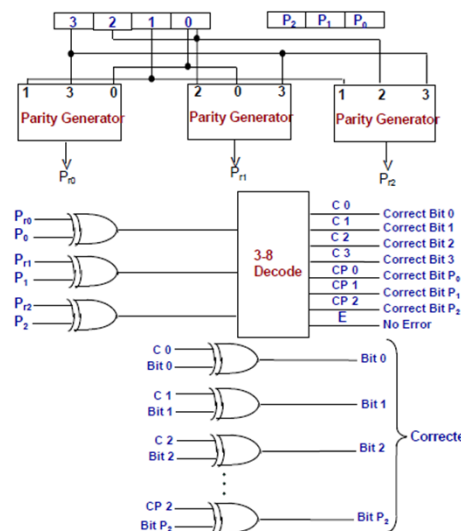
Generalized Overlapping Parity Codes

- The previous slide showed how to use overlapping parity to detect and diagnose single-bit errors
- For single-bit errors, there are k+r possible errors
    - Therefore, we need $2^r \geq k + r + 1$ to uniquely diagnose errors
- In general, can extend this scheme to detect and diagnose more than single-bit errors
    - General approach called "Hamming Codes"

Hamming Error-Correcting Code

- Require from 10% to 40% redundancy
- Best thought of as overlapping parity
- The Hamming single-error correcting code uses c parity check bits to protect k bits of information:
    - $2^c \geq c + k + 1$
- Example:
    - suppose four information bits (d3, d2, d1, d0) and as a result three parity bits (p1, p2, p3)
    - the bits are partitioned into groups as (d3, d1, d0, p1), (d3, d2, d0, p2) and (d3, d2, d1, p3)
        - the grouping of bits can be determine from a list of binary numbers from 0 to 2^k - 1.
    - each check bit is specified to set the parity, either even or odd, of its respective group

7	6	5	4	3	2	1
d_3	d_2	d_1	p_3	d_0	p_2	p_1
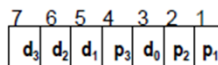
Hamming Error-Correcting Code

Determining the bit groups
(three parity bits)

```
0 0 0
0 0 1     1
0 1 0          2
0 1 1     3    3
1 0 0              4
1 0 1     5        5
1 1 0          6    6
1 1 1     7    7    7
```

Parity bits calculation
p1 = XOR of bits (3, 5, 7)
p2 = XOR of bits (3, 6, 7)
p3 = XOR of bits (5, 6, 7)

Parity checking
c1 = XOR of bits (1,3, 5, 7)
c2 = XOR of bits (2,3, 6, 7)
c3 = XOR of bits (4,5, 6, 7)

■ Observe that each group of bits for parity checking starts with a number that is a power of 2, e.g., 1, 2, 4.

(7,4) Hamming Code

■ Class of (n,k) Hamming codes, e.g., (7,4) [r= n-k =3]
■ Let i1, i2, i3, i4 be the information bits
■ Let p1, p2, p4 be the check bits
■ p1 = i1 xor i2 xor i4
■ p2 = i1 xor i3 xor i4
■ p4 = i2 xor i3 xor i4
■ Let <u>H</u> be the Parity Check Matrix
■ If <u>C</u> is a codeword, then <u>H</u> <u>C</u> = <u>0</u> (mult modulo 2!)
■ Else, <u>H</u> <u>C</u> = <u>S</u>, where <u>S</u> is the syndrome
   ■ Syndrome identifies where error occurred (i.e., which bit)
   ■ This works out like magic because of some cute math

(7,4) Hamming Code

- $\underline{H}$ =

p1	p2	i1	p4	i2	i3	i4
1	0	1	0	1	0	1
0	1	1	0	0	1	1
0	0	0	1	1	1	1

- Info word: 0101: p1 = 0, p2 = 1, p4 = 0
  - codeword is 0100101
- Example1:
  - received error-free codeword R = 0100101
  - Compute syndrome: $\underline{S} = \underline{H}\,\underline{R} = \underline{0} = [0\ 0\ 0]$
- Example 2:
  - received R = 0110101 (i.e., error in bit position 3)
  - Compute syndrome: S = H R = [1 1 0]
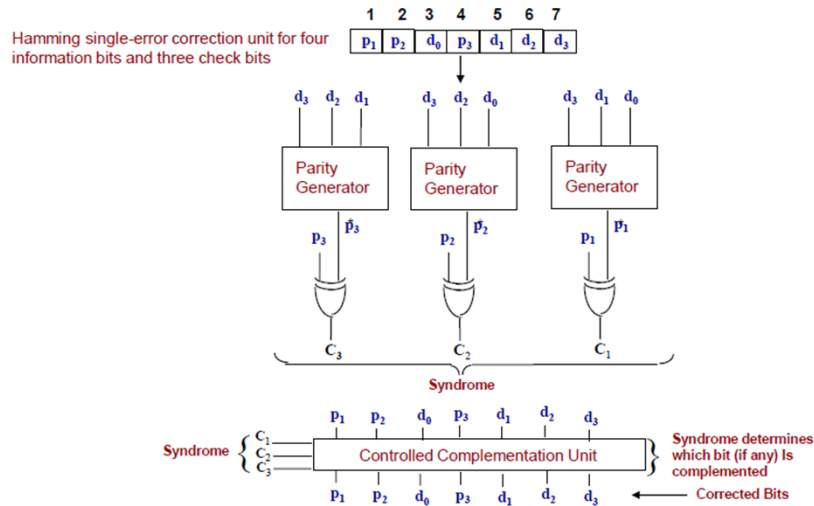   - Read backwards this is 011 = 3

Check Bits and Syndromes for Single-Bit Errors

- The original data is encoded by generating a set C_g, of parity bits.
- To check correctness, the encoding process is repeated and a set C_c, of parity bits is generated.
- If C_g and C_c agree, the information is correct.
- If C_g and C_c disagree, the information is incorrect and must be corrected.
- To aid the correction, a *syndrome* is defined:
  - The syndrome is a binary word that has 1 in each bit position in which C_g and C_c disagree; the syndrome points directly to the erroneous bit.

Erroneous bits	Check bits affected	Syndromes
d_0	p_1, p_2	110
d_1	p_1, p_3	101
d_2	p_2, p_3	011
d_3	p_1, p_2, p_3	111
p_1	p_1	100
p_2	p_2	010
p_3	p_3	001

KIT – University of the State of Baden-Wuerttemberg and
National Laboratory of the Helmholtz Association

Hamming Single-Error Correction Unit

Hamming single-error correction unit for four information bits and three check bits

Single Error Correction and Double Error Detection Hamming Code (SEC-DED)

p_1	p_2	d_0	p_3	d_1	d_2	d_3	p_4
1	1	1	1	1	1	1	1
1	0	1	0	1	0	1	0
0	1	1	0	0	1	1	0
0	0	0	1	1	1	1	0

c_1	c_2	c_3	c_4	
0	0	0	0	No errors
x_3	x_2	x_1	1	Single error in a position ($x_1 x_2 x_3$)
y_3	y_2	y_1	0	Double error
0	0	0	1	Error in bit p_4

- Consider a data word consisting of four information bits
- Three parity bits are needed to provide single error correction
- Adding an extra parity bit, the Hamming code can be used to **correct single bit errors and to detect double errors**

Check bits computation
$P_1 = XOR (3, 5, 7)$
$P_2 = XOR (3, 6, 7)$
$P_3 = XOR (5, 6, 7)$
$P_4 = $ parity over the first 7 bits of the code word

Syndromes computation
$C_1 = XOR (1, 3, 5, 7)$
$C_2 = XOR (2, 3, 6, 7)$
$C_3 = XOR (4, 5, 6, 7)$
$C_4 = $ parity over all 8 bits of the code word

KIT – University of the State of Baden-Wuerttemberg and
National Laboratory of the Helmholtz Association

Single Error Correction and Double Error Detection Hamming Code (SEC-DED) Example

	p_1	p_2	d_0	p_3	d_1	d_2	d_3	p_4
1	1	1	0	0	1	1	0	0
2	1	1	(1)	0	1	1	0	0
3	1	1	0	0	1	(0)	0	0
4	(0)	1	0	0	(0)	1	0	0
5	1	1	0	0	1	1	0	(1)

Failure scenarios

Initial daa

d_0 d_1 d_2 d_3
0 1 1 0

	c_1	c_2	c_3	c_4	
1	0	0	0	0	No errors
2	1	1	0	1	Single error in position 3
3	0	1	1	1	Single error in position 6
4	0	0	1	0	Double error
5	0	0	0	1	Error in bit p_4

Corresponding Syndromes